

# OSCILLATIONS ABOUT EQUILIBRIUM

Some definitions for periodic motion:

Period:  $T \equiv$  time required for one cycle of a periodic motion

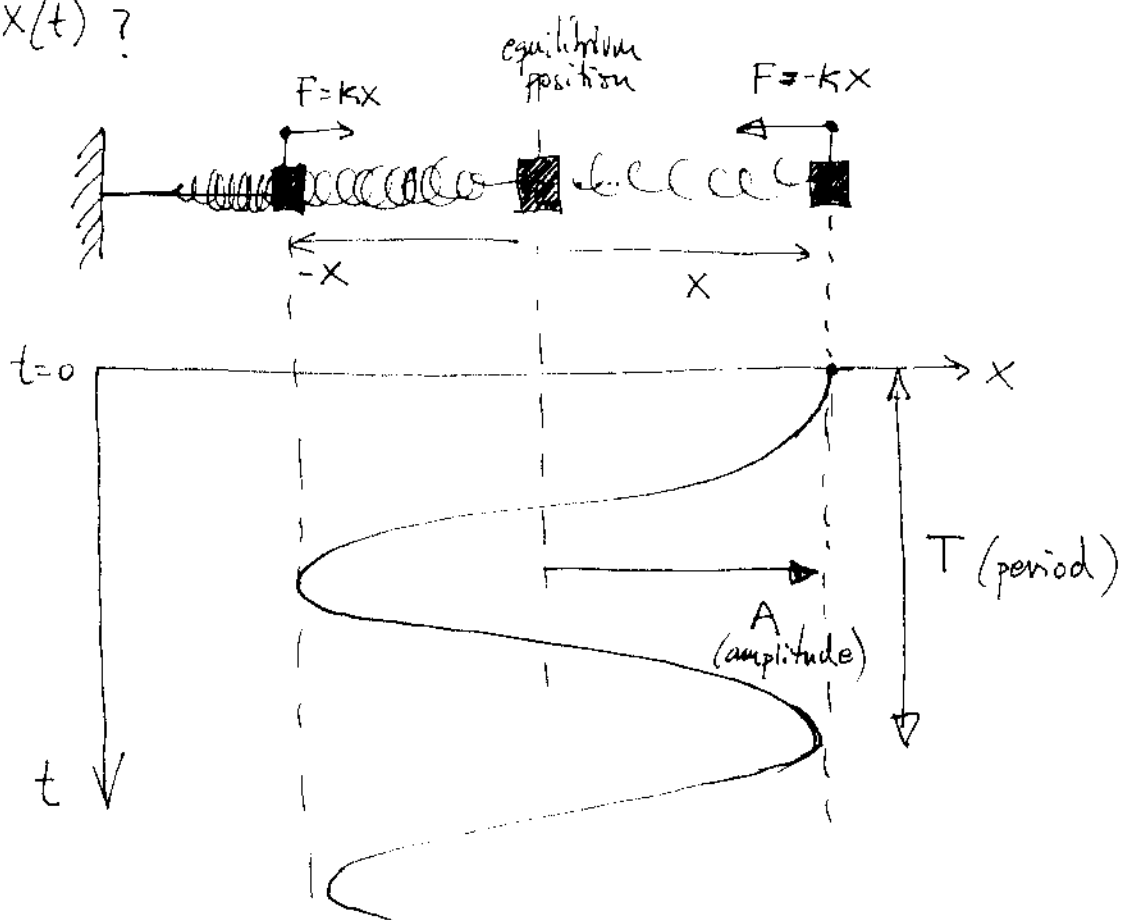
Frequency:  $f = \frac{1}{T}$   $[f] = \frac{1}{s} = \text{Hz}$   $1 \text{ Hz} = 1 \text{ cycle/second}$

Circular periodic motion:

Frequency (angular):  $\omega = 2\pi f$   $[\omega] = \frac{\text{rad}}{\text{second}} = \frac{1}{s} = \text{Hz}/2\pi$

How to describe the periodic motion of an object?

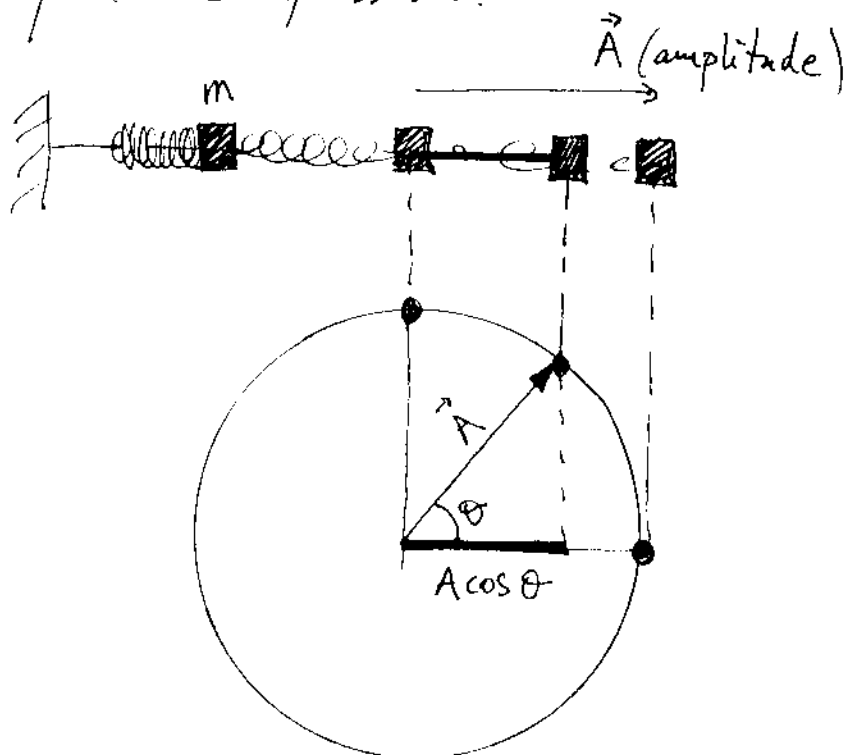
$x(t)$ ?



What's the expression of "x" as a function of time?

→ We need calculus to find it.

Hint: We can use what we know about circular motion to find the expression.



this means that the position of the mass "m" is given by

$$x = A \cos \theta$$

We know that in rotational motion,  $\theta = \omega t$ , so:

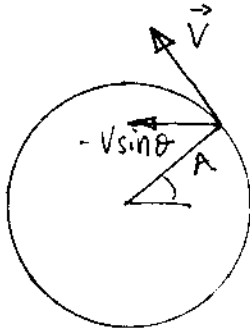
$$x = A \cos(\omega t) = A \cos(2\pi f t) = A \cos\left(\frac{2\pi}{T} t\right)$$

\* 1<sup>st</sup>. equation of periodic motion for position.

$$x_{\text{MAX}} = +A \quad \left[ \cos(\omega t) = +1 \Rightarrow \omega t = n\pi \Rightarrow t = \frac{n\pi}{\omega} = \frac{n\pi}{2\pi} T = \frac{n}{2} T \right]$$

$$x_{\text{MIN}} = 0 \quad \left[ \cos(\omega t) = 0 \Rightarrow \omega t = (2n+1)\frac{\pi}{2} \Rightarrow t = n\frac{T}{4} \right]$$

What about the velocity?



$$V_x = -V \sin \theta = -A\omega \sin(\omega t)$$

$$V = r\omega$$

$$V = -A\omega \sin(\omega t)$$

$$V_{\text{MAX}} = -A\omega \quad [\sin(\omega t) = \pm 1]$$

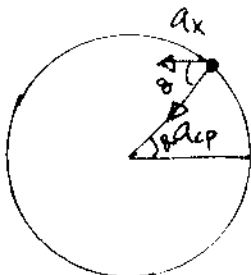
2<sup>nd</sup>. equation of periodic motion for velocity.

$$V_{\text{MAX}}: [\sin(\omega t) = \pm 1 \Rightarrow \omega t = (2n+1)\frac{\pi}{2} \Rightarrow t = \frac{(2n+1)\pi}{2\omega}]$$

$$\Rightarrow t = (2n+1)\frac{\pi}{2} \frac{T}{2\pi} = (2n+1)\frac{T}{4}$$

$$V_{\text{MIN}} = 0 \quad [\sin(\omega t) = 0 \Rightarrow \omega t = n\pi \Rightarrow t = n\frac{T}{2}]$$

What about the acceleration?



$$a_{cp} = r\omega^2$$

$$a_x = -a_{cp} \cos \theta = -r\omega^2 \cos(\omega t)$$

$$a = -A\omega^2 \cos(\omega t)$$

$$a_{\text{MAX}} = -A\omega^2 \quad [\cos(\omega t) = \pm 1, t = n\frac{T}{2}]$$

$$a_{\text{MIN}} = 0 \quad [\cos(\omega t) = 0, t = n\frac{T}{4}]$$

# Summary of equations of periodic motion

$$X = A \cos(\omega t) = A \cos(2\pi f t) = A \cos\left(\frac{2\pi}{T} t\right)$$

$$V = -A\omega \sin(\omega t)$$

$$a = -A\omega^2 \cos(\omega t)$$

$$X_M = A \rightarrow t = n\frac{T}{2}$$

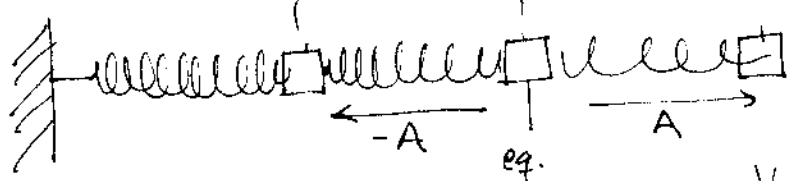
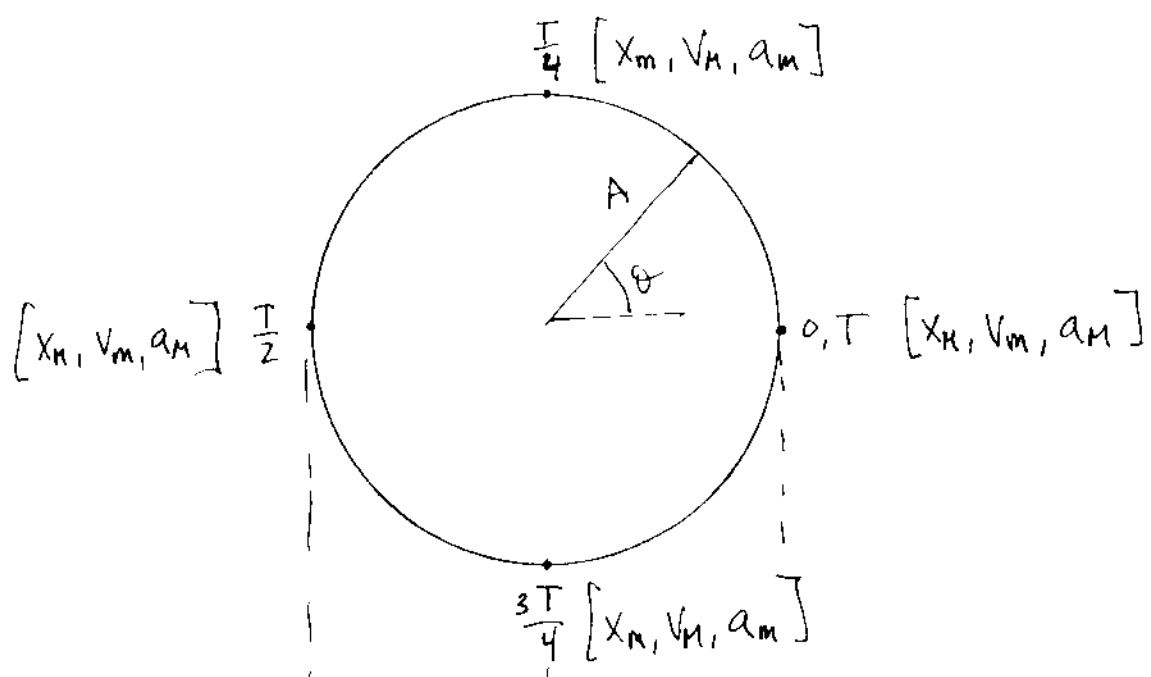
$$V_M = \pm A\omega \rightarrow t = (2n+1)\frac{T}{4}$$

$$a_M = \pm A\omega^2 \rightarrow t = n\frac{T}{2}$$

$$X_M = 0 \rightarrow t = (2n+1)\frac{T}{4}$$

$$V_M = 0 \rightarrow t = n\frac{T}{2}$$

$$a_M = 0 \rightarrow t = (2n+1)\frac{T}{4}$$



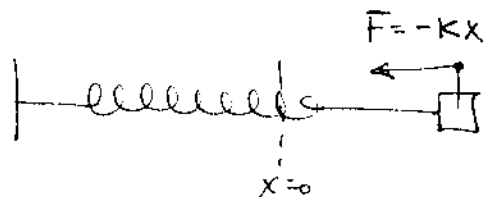
$$\begin{aligned} V_M &= 0 \\ X_M &= -A \\ a_M &= \frac{kx}{m} \end{aligned}$$

$$\begin{aligned} V_M & \\ X_M & (x=0) \\ a_M & (F=0) \end{aligned}$$

$$\begin{aligned} V_M &= 0 \\ X_M &= A \\ a_M &= -\frac{kx}{m} \end{aligned}$$

HOW TO FIND THE PERIOD IN SEVERAL SITUATIONS ?

(A) THE PERIOD OF A MASS ON A SPRING



$$F = -kx = ma$$

we have seen that  $a = -A\omega^2 \cos(\omega t)$ , we substitute this in the previous formula and we obtain ;

$$m[-A\omega^2 \cos(\omega t)] = -kx$$

we know  $x = A \cos(\omega t)$ , then ;

$$m[-A\omega^2 \cos(\omega t)] = -k[A \cos(\omega t)]$$

we obtain

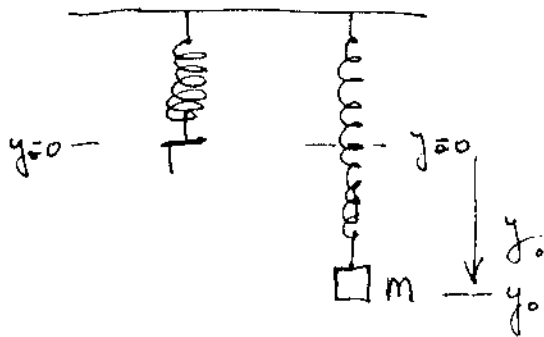
$$m \cancel{A} \omega^2 = k \cancel{A} \rightarrow \omega^2 = \frac{k}{m} \rightarrow \boxed{\omega = \sqrt{\frac{k}{m}}}$$

Since  $T = 2\pi \omega$  then ;

$$\boxed{T = 2\pi \sqrt{\frac{m}{k}}}$$

Increasing  $m$  or decreasing  $k$   
the period will decrease

(A1) PARTICULAR CASE = VERTICAL SPRING



$$\rightarrow mg = ky_0$$

$$\text{so } \boxed{y_0 = \frac{m}{k}g}$$

New equilibrium position

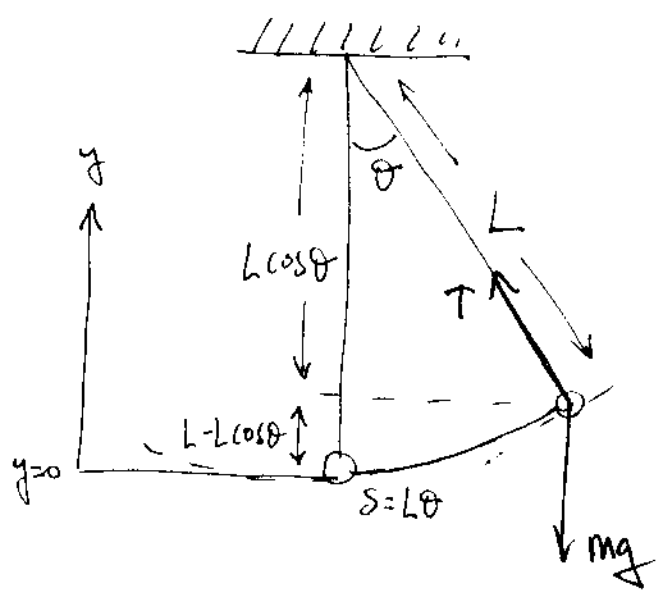
$$\text{We know that } T = 2\pi\sqrt{\frac{m}{k}} \rightarrow \frac{k}{m} = \left(\frac{T}{2\pi}\right)^2 \rightarrow \frac{m}{k} = \left(\frac{2\pi}{T}\right)^2$$

then the new eq. position is related  $T$  through:

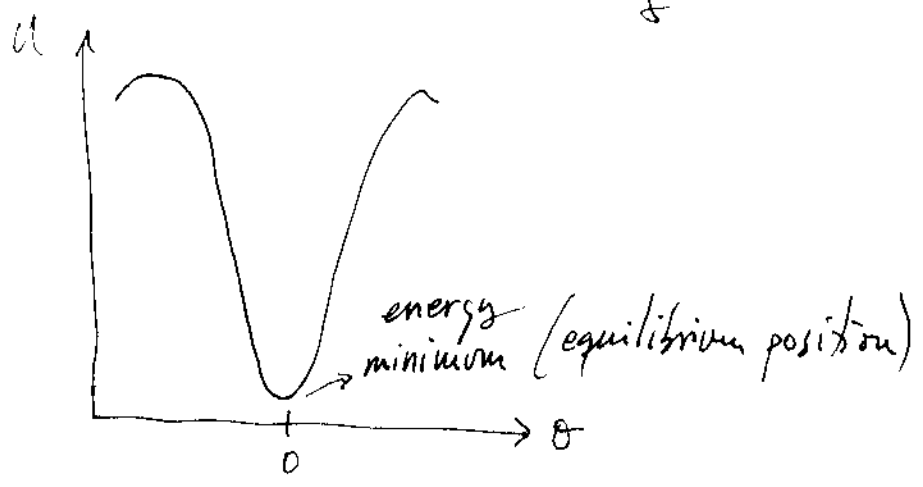
$$\boxed{y_0 = \frac{m}{k}g = \left(\frac{2\pi}{T}\right)^2 g}$$

But the period of oscillation will be the same.

# (B) THE PERIOD OF A PENDULUM



Potential energy:  $U = mg(L - L \cos \theta) = mgL(1 - \cos \theta)$



Tangential force always points to the equilibrium position.

$$F = mg \sin \theta$$

The arc covered ~~over~~  $\theta$ , is  $s = L\theta$

For small angles, we can approximate  $\sin \theta \approx \theta$ . Then:

$$F = mg \theta$$

But we know,  $\theta = s/L$ , so:

$$F = mg s/L = \left(\frac{mg}{L}\right) s$$

This eq. has the same form than the spring force:

$$F = kx$$

So the equivalence gives us the spring constant of a pendulum which should be called, the restoring force constant:

$$k = \frac{mg}{L} \rightarrow \frac{m}{k} = \frac{L}{g}$$

finally, the period of a pendulum (for small angle oscillations)

is:

$$T = 2\pi \sqrt{\frac{L}{g}}$$

if  $g$  increases  $T$  decreases (faster oscillation in Earth than Moon)

if  $L$  increases  $T$  increases (~~faster~~ faster oscill. for shorter pend.)