ROTATIONAL MOTION

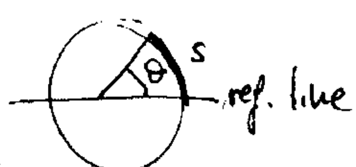
this chapter deals with rotational motion and all definitions and applications are equivalent to what we have seen in chapter 2 (1-D kinematics) that we should now call: "translational kinematics" to differenciate from "rotational kinematics".

As we will see, the equations of rotational motion have the same form than the equations of translational motion, by just changing "x" by "θ", "v" by "ω" and "a" by "α", where:

ANGULAR POSITION, VELOCITY AND ACCELERATION

1) θ is the angle measured from a reference line

$[\theta] = \text{rad}$



$\boxed{\theta > 0 \text{ counterclockwise}}$

and "s" is the arc length

Note that $s = r\theta$

* so 1 radian is the angle for which the arc length "s" on a circle of radius "r" is equal to the radius.
* One revolution, $1\,\text{rev} = 2\pi\,\text{rad}$.
* [rad] is dimensionless.

We define angular displacement as: $\Delta\theta = \theta_f - \theta_i$

2) <u>ω is the angular velocity, and is defined as:</u>

$$\omega_{av} = \frac{\Delta\theta}{\Delta t} \qquad [\omega_{av}] = s^{-1} \equiv \frac{rad}{s}$$

in the same way we did with linear velocity "v" we can also define instantaneous angular velocity "ω" as;

$$\omega = \lim_{\Delta t \to 0} \frac{\Delta\theta}{\Delta t} = \frac{d\theta}{dt}$$

$\omega > 0$ counterclockwise rotation

<u>time to complete a revolution (Period)</u>

$$\omega = \frac{\Delta\theta}{\Delta t} = \frac{2\pi}{T} \longrightarrow T = \frac{2\pi}{\omega} \qquad [T] = s$$

3) <u>α is the angular acceleration.</u>

$$\alpha_{av} = \frac{\Delta\omega}{\Delta t}$$ and instantaneous angular acceleration

$$\alpha = \lim_{\Delta t \to 0} \frac{\Delta\omega}{\Delta t} \qquad [\alpha] = \frac{rad}{s^2} = s^{-2}$$

$$\alpha = \frac{d\omega}{dt}$$

EQUATIONS OF ROTATIONAL MOTION

Once we have found the equivalence between positions, velocities and accelerations, we can obtain the equations of motion for rotation by direct translation of the equations of translational motion seen in chapter 2.

As a reminder (for constant a, α)

$$\left.\begin{aligned} x &= x_0 + v_0 t + \tfrac{1}{2} a t^2 \\ v &= v_0 + a t \end{aligned}\right\} \text{Translational motion}$$

$$\left.\begin{aligned} \theta &= \theta_0 + \omega_0 t + \tfrac{1}{2} \alpha t^2 \\ \omega &= \omega_0 + \alpha t \end{aligned}\right\} \text{Rotational motion}$$

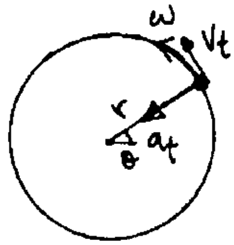
ANGULAR AND TRANSLATIONAL QUANTITIES

<u>Tangential velocity</u> <u>v_t</u>:

$$v_t = \frac{\Delta x}{\Delta t} = \frac{\text{perimeter}}{1\ \text{period}} = \frac{2\pi r}{T} = \frac{2\pi r}{\cancel{2\pi}/\omega} = r\omega$$

so $$\boxed{v_t = r\omega = r\,\frac{2\pi}{T}}$$

Centripetal accelerations:



We have seen in previous chapters that,

$$a_{cp} = \frac{v_t^2}{r}$$

So in terms of $\omega = \frac{v_t}{r}$ we have;

$$a_{cp} = \frac{(r\omega)^2}{r} = r\omega^2 \qquad [a_{cp}] = m/s^2$$

$$\boxed{a_{cp} = \frac{v_t^2}{r} = r\omega^2 = r\left(\frac{2\pi}{T}\right)^2}$$

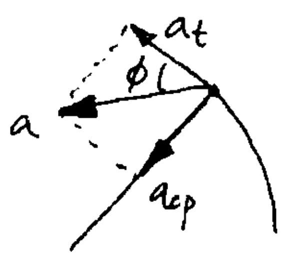
due to changing direction

Tangential acceleration:

$$a_t = \frac{\Delta v_t}{\Delta t} = r\,\frac{\Delta\omega}{\Delta t} = r\alpha \qquad \boxed{a_t = r\alpha}$$

due to change of speed

Relation between centripetal and tangential accelerations



$$a = \sqrt{a_t^2 + a_{cp}^2}$$

$$\phi = \tan^{-1}\left(\frac{a_{cp}}{a_t}\right)$$

ROTATIONAL KINETIC ENERGY

We have seen in previous chapters that linear kinetic energy (or translational) is on the form;

$$K = \frac{1}{2} m V^2$$

We know that $v_t = r\omega$, therefore, the rotational kinetic energy is given by;

$$K_R = \frac{1}{2} m V^2 = \frac{1}{2} m (r\omega)^2$$

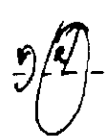
We choose to write this in the next form,

$$K_R = \frac{1}{2} m r^2 \omega^2 = \frac{1}{2} I \omega^2$$

where $I = mr^2$ is the moment of inertia of an objet.

In general $I = \sum m_i r_i^2$. The moment of inertia of an object is therefore dependent on its geometry and the axis of rotation.

HOOP



$I = MR^2$

DISK

$I = \frac{1}{2} MR^2$

<u>Kinetic energy of rolling motion:</u>

$$K = K + K_R = \frac{1}{2} m V^2 + \frac{1}{2} I \omega^2$$

CALCULATION OF MOMENTS OF INERTIA

For multiple particles, $I = \sum_i r_i^2 m_i$

For solid/continuous objects:

$$I = \lim_{\Delta m_i \to 0} \sum_i r_i^2 \Delta m_i = \int r^2 dm = \int \rho r^2 dV$$

where $\rho = m/V$

If the object is homogeneous ρ = constant, and then

$$I = \rho \int r^2 dV$$

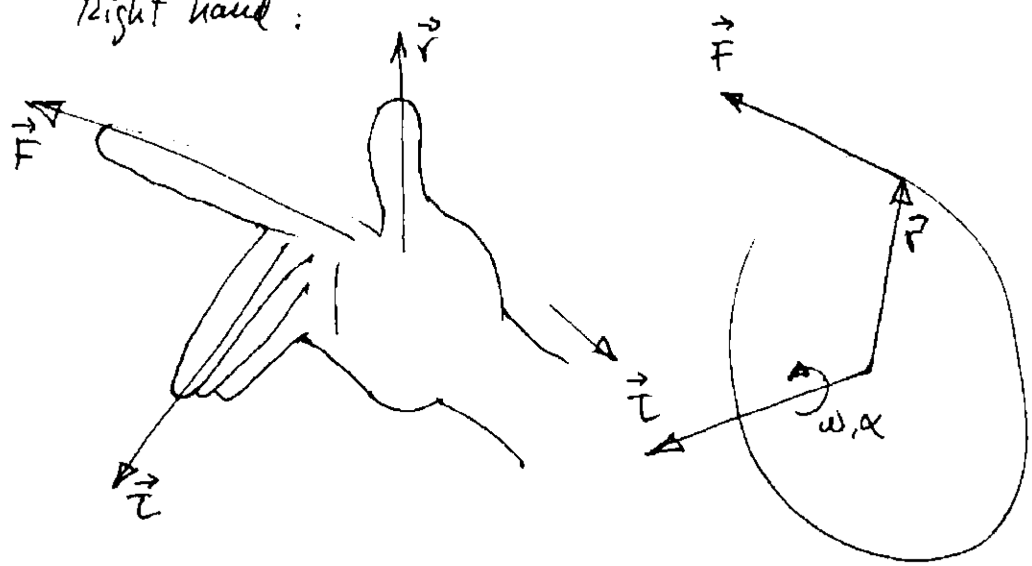
TORQUE

$\vec{\tau} = \vec{r} \times \vec{F}$ $[\tau] = N \cdot m$

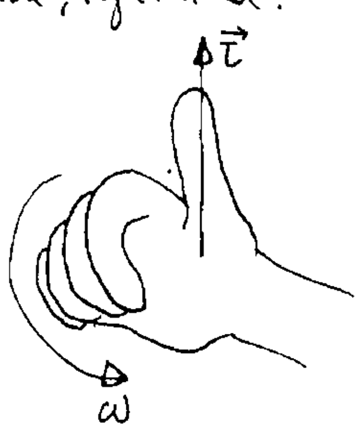
Note that the cross product of two vectors is another vector.

Direction of the torque.

Right hand:



Motion, right hand:

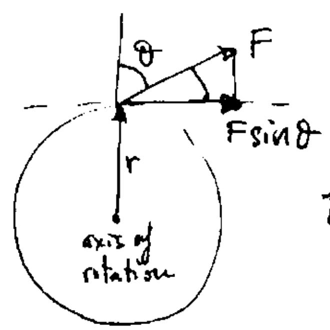


If we remember Work done by a force $W = \vec{F}\cdot\vec{d} = Fd\cos\theta$
where only the force component on the direction of the displacement contributed to the work (scalar),

with torque we have something different.

$$\tau = rF\sin\theta$$

only the force component perpendicular to a radial direction, this is tangential force, contributes to the torque.



$$\tau = rF\sin\theta$$

By convention, $\tau > 0$ if torque causes counterclockwise rotation
$\tau < 0$ " clockwise "

MOTION UNDER A NET TORQUE

2nd Newton's law: $\sum F = ma$

$$\sum \tau = \sum F r \underset{\sum F = ma}{=} (ma) r \underset{a = r\alpha}{=} (m r \alpha) r = (m r^2) \alpha = I \alpha$$

$\boxed{\sum \tau = I \alpha}$ 2nd Newton's law for rotation

ANGULAR MOMENTUM

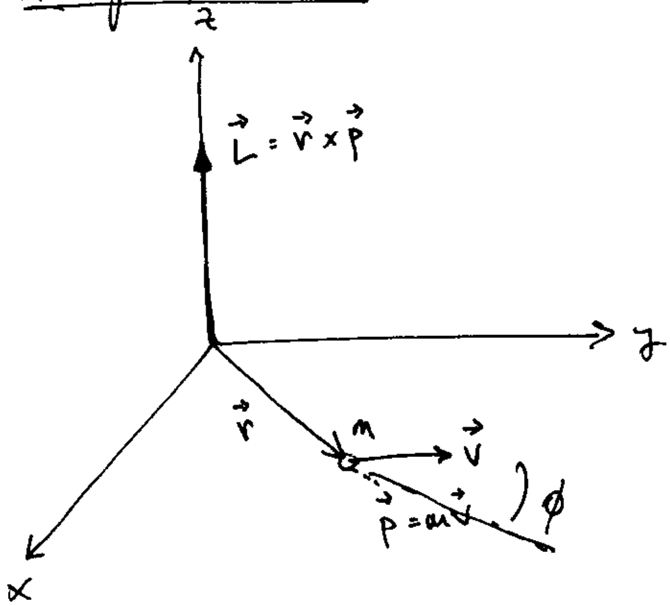
Definition:

$$\vec{L} \equiv \vec{r} \times \vec{p} \quad \sim \quad \left(\vec{p} = m\vec{v} \right)$$

And, therefore:

$$\sum \vec{\tau} = \frac{d\vec{L}}{dt} \quad \sim \quad \left(\sum \vec{F} = \frac{d\vec{p}}{dt} \right)$$

<u>A single particle:</u>



$$L = rp\sin\phi = rmv\sin\phi$$

$$[L] \equiv \mathrm{kg}\, \mathrm{m}^2/\mathrm{s}$$

<u>A system of particles:</u>

$$\vec{L}_{TOT} = \sum_i \vec{L}_i \Rightarrow \sum_i \vec{\tau}_i = \frac{d\vec{L}_{TOT}}{dt} = \sum \tau_N^{ext}$$

$$\sum \vec{\tau}_i = \underbrace{\cancel{\sum \vec{\tau}_i}}_{\text{INTERNAL} \to 0} + \underbrace{\sum \vec{\tau}_i}_{\text{EXTERNAL}}$$

ANGULAR MOMENTUM OF A ROTATING OBJECT

$$L = rp \underset{\substack{\uparrow \\ p = mv}}{=} rmv \underset{\substack{\uparrow \\ v = r\omega}}{=} mr^2\omega \underset{\substack{\uparrow \\ I = mr^2}}{=} I\omega$$

$$\boxed{\vec{L} = I\vec{\omega}}$$

Therefore:

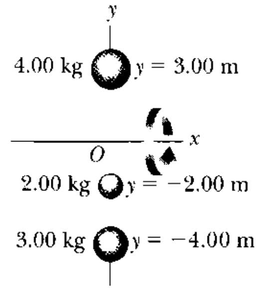
$$\boxed{\frac{d\vec{L}}{dt} = I\frac{d\vec{\omega}}{dt} = I\alpha = \sum \tau_{ext}}$$

CONSERVATION OF ANGULAR MOMENTUM

If the net external torque acting on a system is zero, then the angular momentum is conserved

$$\sum \vec{\tau}_{ext} = 0 = \frac{d\vec{L}}{dt} = 0 \implies \vec{L} = \text{constant}.$$

● Rigid rods of negligible mass lying along the y axis connect three particles (Fig. P10.22). The system rotates about the x axis with an angular speed of 2.00 rad/s. Find (a) the moment of inertia about the x axis and the total rotational kinetic energy evaluated from $\frac{1}{2}I\omega^2$ and (b) the tangential speed of each particle and the total kinetic energy evaluated from $\Sigma\ \frac{1}{2}m_i v_i^2$. (c) Compare the answers for kinetic energy in parts (a) and (b).

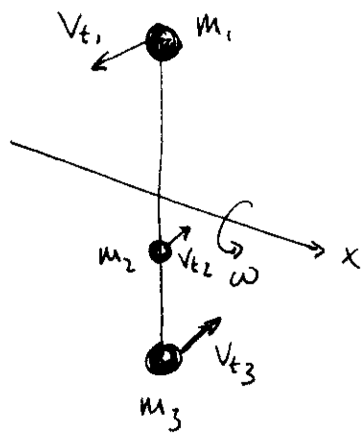


$\omega = 2$ rad/s

a) I_x and K_R ?

b) V_{t_i} ? $K_R(v_t)$?

c) compare



$m_1 = 4$ kg $\quad y_1 = 3$ m
$m_2 = 2$ kg $\quad y_2 = -2$ m
$m_3 = 3$ kg $\quad y_3 = -4$ m

a) $I_x = m_1 y_1^2 + m_2 y_2^2 + m_3 y_3^2 = 92 \text{ kg}\cdot\text{m}^2$

$K_R = \frac{1}{2} I_x \omega^2 = \frac{1}{2}\, 92 \text{ kg}\cdot\text{m}^2 \left(2 \frac{\text{rad}}{s}\right)^2 = 184$ J

b) $V_1 = r_1 \omega = 3\text{m} \cdot 2\frac{\text{rad}}{s} = 6$ m/s

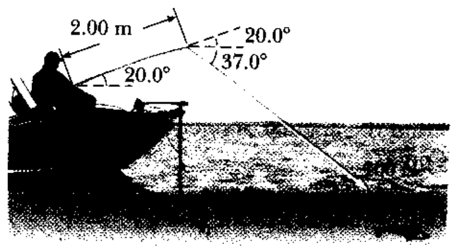
$V_2 = r_2 \omega = 4$ m/s

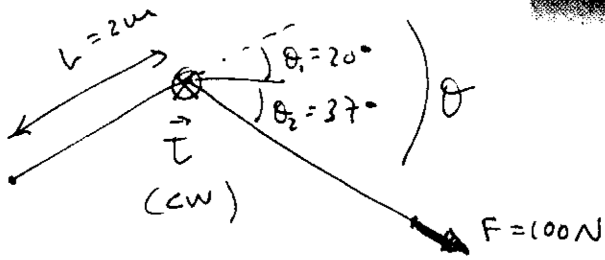
$V_3 = r_3 \omega = 8$ m/s

$K = K_1 + K_2 + K_3 = \frac{1}{2} m_1 V_1^2 + \frac{1}{2} m_2 V_2^2 + \frac{1}{2} m_3 V_3^2 = 184$ J

c) $K_R = K_R$! what a surprise !

The fishing pole in Figure P10.32 makes an angle of 20.0° with the horizontal. What is the torque exerted by the fish about an axis perpendicular to the page and passing through the angler's hand?



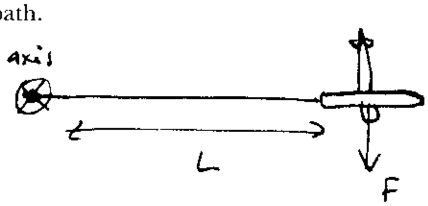


$$\vec{\tau} = \vec{r} \times \vec{F} = rF\sin\theta = 2\,m \cdot 100\,N \cdot \sin 57° = 168\ N\cdot m > 0\ (cw)$$

▲ A model airplane with mass 0.750 kg is tethered by a wire so that it flies in a circle 30.0 m in radius. The airplane engine provides a net thrust of 0.800 N perpendicular to the tethering wire. (a) Find the torque the net thrust produces about the center of the circle. (b) Find the angular acceleration of the airplane when it is in level flight. (c) Find the translational acceleration of the airplane tangent to its flight path.

$L = 30\,m$

$F = 0.8\,N$



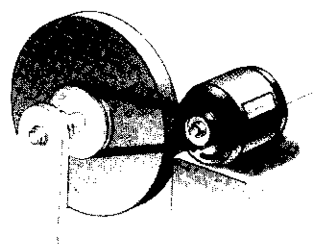
a) $\vec{\tau} = \vec{r} \times \vec{F} = rF\underbrace{\sin 90°}_{+1} = rF$ (cw)

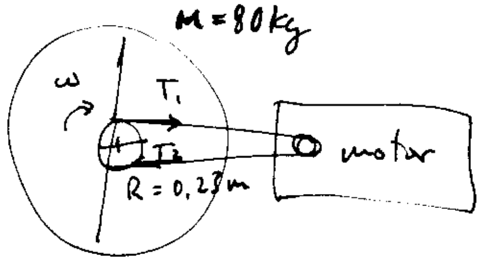
$$\boxed{\vec{\tau} = 30\,m\,(0.8\,N) = 24\ N\cdot m}$$

b) $$\alpha = \frac{\tau}{I} = \frac{rF}{mr^2} = \frac{24\ Nm}{0.75\,kg \cdot (30)^2\,m^2} = 0.035\ rad/s^2$$

c) $a_t = r\alpha = 1.07\ m/s^2$

An electric motor turns a flywheel through a drive belt that joins a pulley on the motor and a pulley that is rigidly attached to the flywheel as shown in Figure P10.39. The flywheel is a solid disk with a mass of 80.0 kg and a diameter of 1.25 m. It turns on a frictionless axle. Its pulley has much smaller mass and a radius of 0.230 m. The tension in the upper (taut) segment of the belt is 135 N, and the flywheel has a clockwise angular acceleration of 1.67 rad/s². Find the tension in the lower (slack) segment of the belt.





$T_1 = 135N$

$\alpha = 1.67\ rad/s^2$

$\varnothing = 1.25m$

$$\vec{\tau}_1 = \vec{r}_1 \times \vec{F}_1 = -RT_1 \underbrace{\sin 90°}_{+1}$$

$$\vec{\tau}_2 = \vec{r}_2 \times \vec{F}_2 = RT_2$$

$$\sum \vec{\tau} = I\vec{\alpha}$$

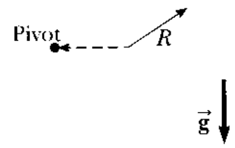
$$\sum \tau = \tau_1 + \tau_2 = R(T_2 - T_1) = \frac{1}{2} m \left(\frac{\varnothing}{2}\right)^2 \cdot \alpha$$

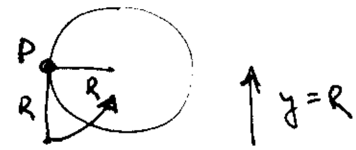
$$\boxed{T_2 = \frac{m\varnothing^2}{4R}\alpha + T_1 = 21.5N}$$

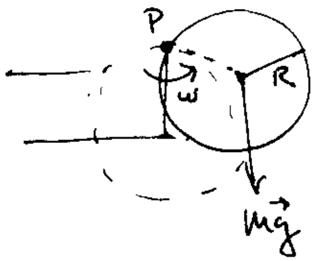
$\boxed{\alpha < 1}$

$\alpha = -1.67 \frac{rad}{s^2}$

(a) A uniform solid disk of radius R and mass M is free to rotate on a frictionless pivot through a point on its rim (Fig. P10.51). If the disk is released from rest in the position shown by the blue circle, what is the speed of its center of mass when the disk reaches the position indicated by the dashed circle? (b) What is the speed of the lowest point on the disk in the dashed position? (c) **What If?** Repeat part (a) using a uniform hoop.

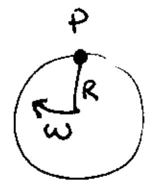






$$E_i = mgR$$

(final)



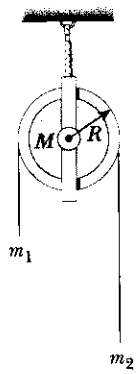
$$E_f = \frac{1}{2} I \omega^2$$

$$\Delta E = 0 \longrightarrow E_f = E_i \longrightarrow \frac{1}{2} I \omega^2 = mgR$$

$$\omega = \sqrt{\frac{2mgR}{I}} = \sqrt{\frac{2mgR}{\frac{3}{2} mR^2}} = \sqrt{\frac{4g}{3R}}$$

$$\boxed{V_{CM} = R\omega = R\sqrt{\frac{4g}{3R}} = 2\sqrt{\frac{Rg}{3}}}$$

● Consider the system shown in Figure P10.44 with $m_1 = 20.0$ kg, $m_2 = 12.5$ kg, $R = 0.200$ m, and the mass of the uniform pulley $M = 5.00$ kg. Object m_2 is resting on the floor, and object m_1 is 4.00 m above the floor when it is released from rest. The pulley axis is frictionless. The cord is light, does not stretch, and does not slip on the pulley. Calculate the time interval required for m_1 to hit the floor. How would your answer change if the pulley were massless?



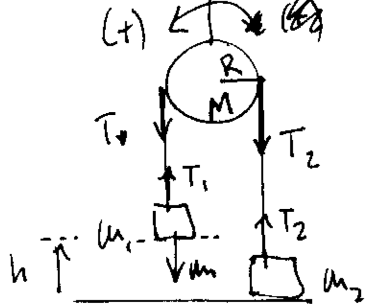
$m_1 = 20$ kg

$m_2 = 12.5$ kg

$R = 0.2$ m

$M = 5$ kg

$h = 4.0$ m



t (m_1 hit floor) ? $\rightarrow a_{m_1}$

$0 = y = h - \frac{1}{2} a t^2$ $\quad \boxed{t = \sqrt{\frac{2h}{a}}}$ we need acceleration:

we know:

$\sum \tau = I\alpha$

$\sum F = ma$

$(m_1)\ -T_1 + m_1 g = m_1 a$

$(m_2)\ T_2 - m_2 g = m_2 a$

$\cancel{R}T_1 - \cancel{R}T_2 = \frac{1}{2} M R^2 \cdot \alpha = \frac{1}{2} M R^2 \frac{a}{R} = \frac{1}{2} M \cancel{R} a$

In addition we know $a = R\alpha$

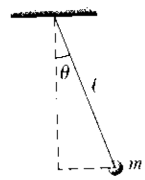
Three equations for three variables:

$$T_1 - T_2 = \frac{1}{2} M a$$
$$+ \left(-T_1 + m_1 g = m_1 a \right)$$
$$+ \left(T_2 - m_2 g = m_2 a \right)$$

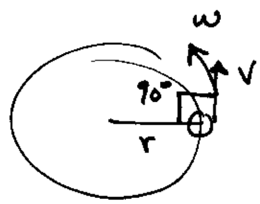
$$(m_1 - m_2) g = (m_1 + m_2) a + \frac{1}{2} M a \; :$$

$$\boxed{a = \frac{(m_1 - m_2) g}{m_1 + m_2 + \frac{1}{2} M} = 2.1 \, m/s^2} \qquad \boxed{t = \sqrt{\frac{2h}{a}} = 1.95 s}$$

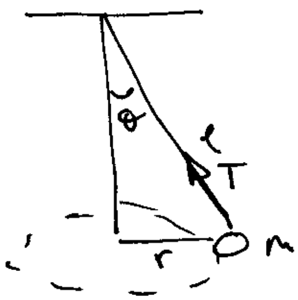
A conical pendulum consists of a bob of mass m in motion in a circular path in a horizontal plane as shown in Figure P11.14. During the motion, the supporting wire of length ℓ maintains the constant angle θ with the vertical. Show that the magnitude of the angular momentum of the bob about the circle's center is



$$L = \left(\frac{m^2 g \ell^3 \sin^4\theta}{\cos\theta}\right)^{1/2}$$



$L = mrv = \vec{r} \times \vec{p} = rp\sin 90°$



$L = rmv$

$\sum F_x = ma_x \qquad T\sin\theta = m a_{cp} = m\frac{v^2}{r}$

$\sum F_y = ma_y = 0 \qquad T\cos\theta - mg = 0$

$\frac{\sin\theta}{\cos\theta} = \frac{v^2}{gr} \qquad v = \sqrt{gr\frac{\sin\theta}{\cos\theta}} = \sqrt{g\ell\frac{\sin^2\theta}{\cos\theta}}$

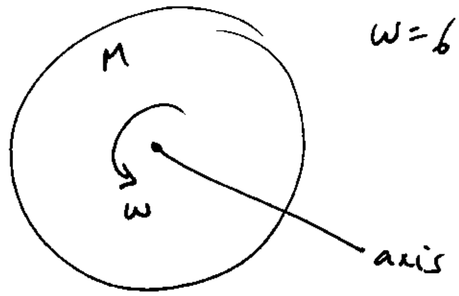
$r = \ell\sin\theta$

$L = rmv = \ell\sin\theta\, m v = \sqrt{m^2 g \ell^3 \frac{\sin^4\theta}{\cos\theta}}$

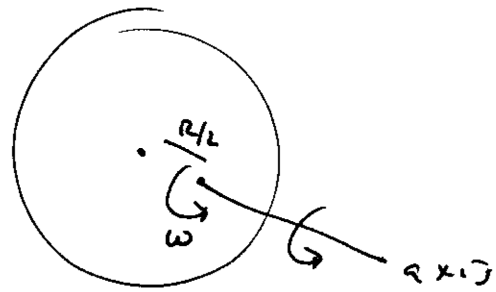
A uniform solid disk of mass 3.00 kg and radius 0.200 m rotates about a fixed axis perpendicular to its face with angular frequency 6.00 rad/s. Calculate the angular momentum of the disk when the axis of rotation (a) passes through its center of mass and (b) passes through a point midway between the center and the rim.

$M = 3\,kg$

$\omega = 6\ rad/s$



a



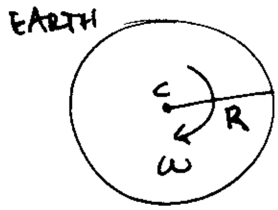
$$L = I\omega = \frac{1}{2}MR^2\omega$$

$$\boxed{L = 0.360\ kg\,m^2/s}$$

$$L = I\omega = \left[\frac{1}{2}MR^2 + M\left(\frac{R}{2}\right)^2\right]\omega$$

$$\boxed{L = 0.54\ kg\cdot m^2/s}$$

● (a) Model the Earth as a uniform sphere. Calculate the angular momentum of the Earth due to its spinning motion about its axis. (b) Calculate the angular momentum of the Earth due to its orbital motion about the Sun. (c) Are the two quantities of angular momentum nearly equal or quite different? Which is larger in magnitude? By what factor?



$M = 5.98 \times 10^{24}$ kg

$R = 6.37 \times 10^{6}$ m

$I = \frac{2}{5} MR^2 = 9.71 \times 10^{37}$ kg m^2

$\omega = \frac{1 \text{ rev}}{24 \text{ h}} \cdot \frac{\text{h}}{3600 \text{ s}} \cdot \frac{2\pi \text{ rad}}{\text{rev}} = 7.27 \times 10^{-5}$ /s

$L = I\omega$

$\boxed{L = 7.06 \times 10^{33} \text{ kg} \cdot \text{m}^2/\text{s}}$

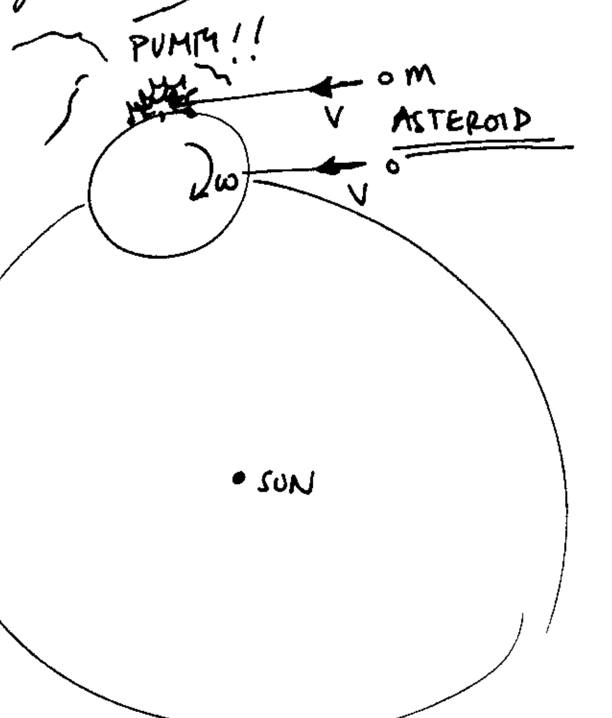
$d = 1.496 \times 10^{11}$ m

$I = Md^2 = 1.34 \times 10^{47}$ kg m^2

$\omega = \frac{1 \text{ rev}}{365 \text{ days}} \cdot \frac{1 \text{ day}}{24 \text{ h}} \cdot \frac{\text{h}}{3600 \text{ s}} \cdot \frac{2\pi}{\text{rev}} = 1.99 \times 10^{-7}$ /s

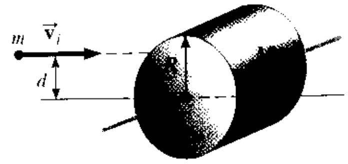
$\boxed{L = I\omega = 2.66 \times 10^{40} \text{ kg} \cdot \text{m}^2/\text{s}}$

orbital angular momentum is much larger

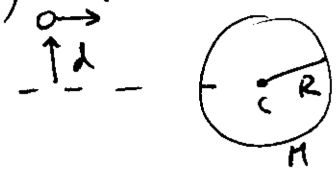


R11.4

● A wad of sticky clay with mass m and velocity $\vec{v}_i$ is fired at a solid cylinder of mass M and radius R (Figure P11.39). The cylinder is initially at rest and is mounted on a fixed horizontal axle that runs through its center of mass. The line of motion of the projectile is perpendicular to the axle and at a distance $d < R$ from the center. (a) Find the angular speed of the system just after the clay strikes and sticks to the surface of the cylinder. (b) Is mechanical energy of the clay-cylinder system conserved in this process? Explain your answer. (c) Is momentum of the clay-cylinder system conserved in this process? Explain your answer.



(initial)



(final)

$\Delta L = 0 \quad L_f = L_i$

$L_i = m\vec{v} \times \vec{d} = mvd$

$L_f = I\omega = \left[\frac{1}{2}MR^2 + mR^2\right]\omega_z$

$L_i = L_f \rightarrow \boxed{\omega = \dfrac{2mvd}{(M+2m)R^2}}$